Econ 802

Answers to First Midterm

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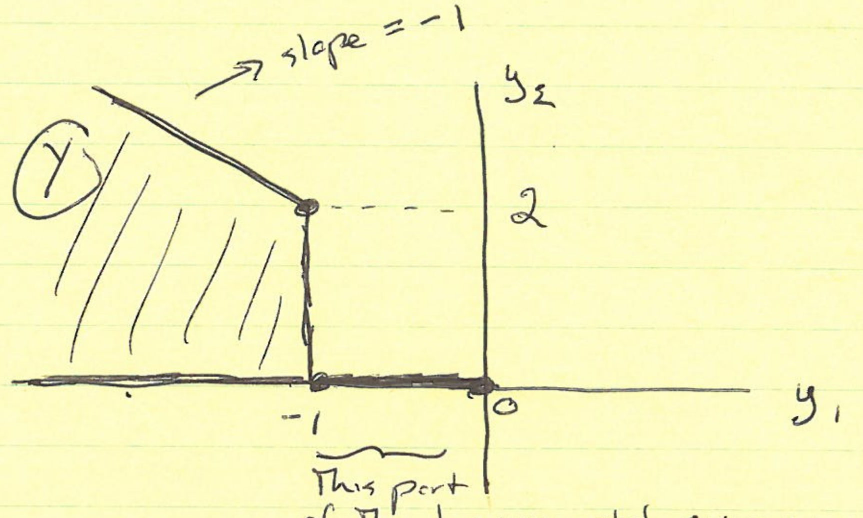
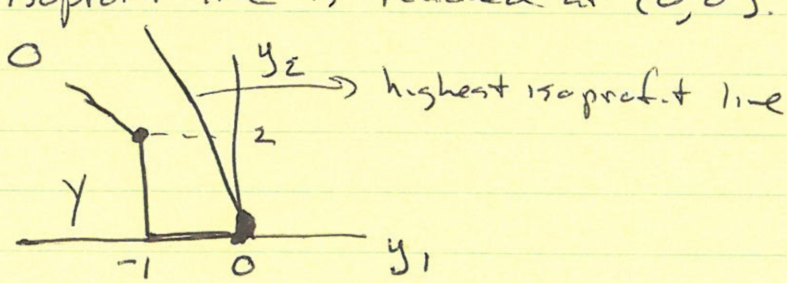
1. (a)



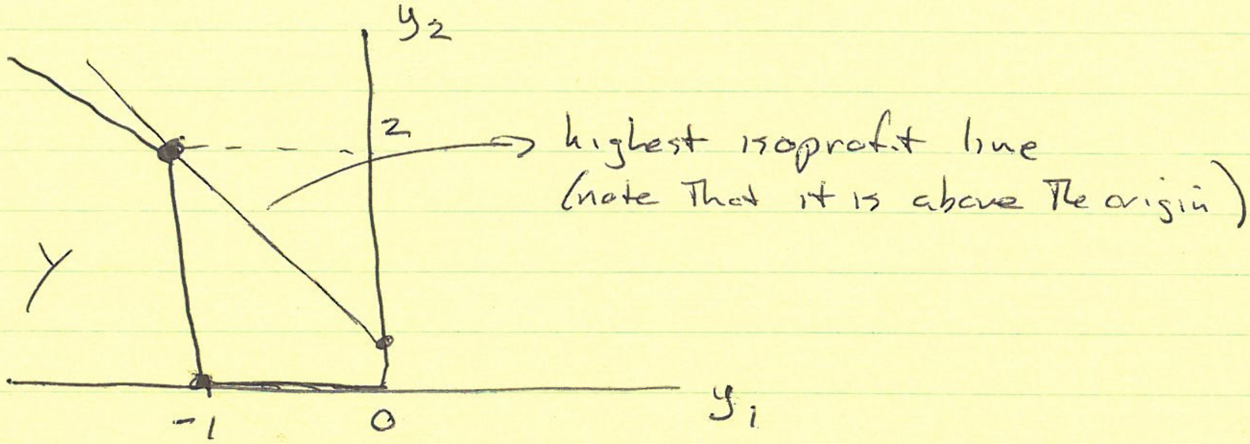
Y is closed because it contains all of its boundary.
Y is not bounded because points arbitrarily far away from the origin are included.
Y is not convex. For instance, points in the interior of the line segment between $(-1, 2)$ and $(0,0)$ are not in Y.

(b) Consider an isoprofit line $\pi = p_1 y_1 + p_2 y_2$.
This has slope $= -\frac{p_1}{p_2}$ $\Rightarrow y_2 = \frac{\pi - p_1 y_1}{p_2}$

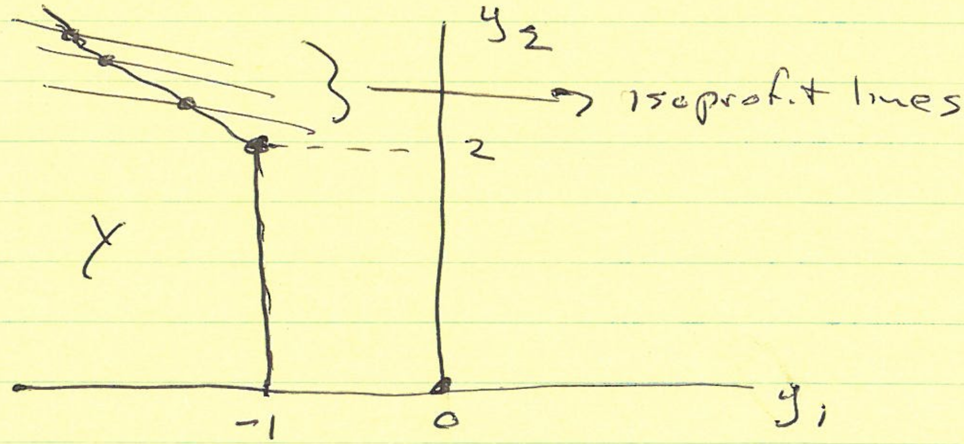
(i) If $\frac{p_1}{p_2} > 2$ the isoprofit lines are steeper than -2 and the highest isoprofit line is reached at $(0,0)$.
Thus $\pi(p_1, p_2) = 0$



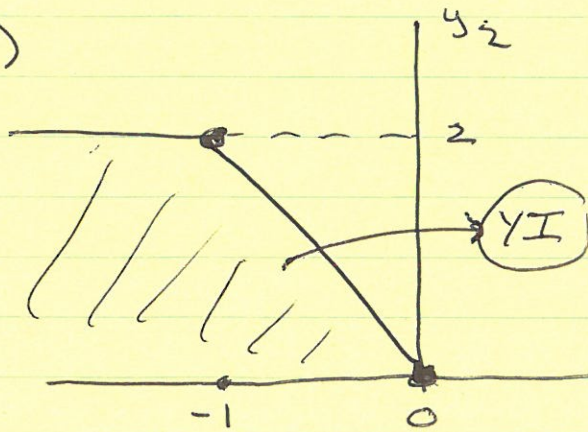
(ii) If $1 \le \frac{p_1}{p_2} \le 2$ the highest isoprofit line is reached at $(-1, 2)$. Thus $\pi(p_1, p_2) = p_1(-1) + p_2(2)$.



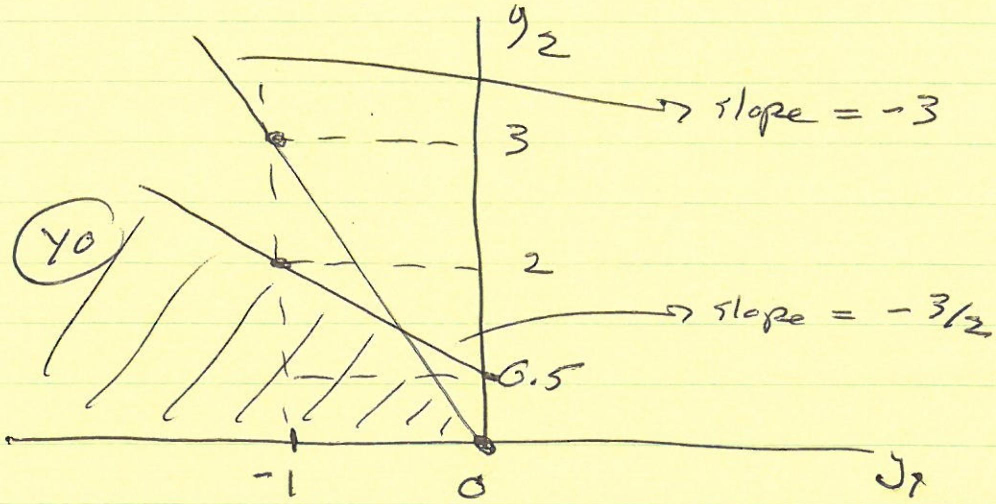
(iii) If $\frac{p_1}{p_2} < 1$ there is no highest isoprofit line; the firm can always get to a higher one by going further out along the boundary of Y to the right. So $\pi(p_1, p_2)$ is not well defined.



1 (c)



This is consistent with observations because $\frac{p_1}{p_2} = 3$ implies isoprofit lines steeper than -2, so highest is at the origin. $\frac{p_1}{p_2} = \frac{3}{2}$ implies isoprofit lines with slope $-\frac{3}{2}$ and the highest isoprofit line goes through $(-1, 2)$.



YO is the set of points below both isoprofit lines.
When $\frac{p_1}{p_2} = 3$, the highest isoprofit line goes through $(0,0)$
When $\frac{p_1}{p_2} = \frac{3}{2}$, the highest isoprofit line goes through $(-1,2)$.

Thus YO is consistent with the observations.

2. (a) Let the initial price vector be (p^1, w) and let the new price vector be (p^2, w) where $p^2 > p^1$. According to WAPM:

$$p^1 y^1 - w x^1 \geq p^1 y^2 - w x^2$$
$$p^2 y^2 - w x^2 \geq p^2 y^1 - w x^1$$

where (y^1, x^1) is the firm's behavior in period 1 and likewise for (y^2, x^2) in period 2

Reorganize:
$$p^1 (y^1 - y^2) \geq w(x^1 - x^2)$$
$$p^2 (y^2 - y^1) \geq w(x^2 - x^1)$$

Sum these inequalities to get $\Delta p \Delta y \geq 0$ where $\Delta p = p^2 - p_1$ and $\Delta y = y^2 - y^1$.

This shows that if output price rises $(\Delta p > 0)$ then output quantity cannot fall $(\Delta y \geq 0)$.

2 (b) Hotelling's Lemma says $\dfrac{\partial \pi(p,w)}{\partial p} = y(p,w)$

and $\dfrac{\partial \pi(p,w)}{\partial w_i} = x_i(p,w)$, $i = 1, 2$.

We know that the profit function is convex so its Hessian matrix is positive semidefinite. Thus

$$\begin{bmatrix} \dfrac{\partial^2 \pi}{\partial p^2} & \dfrac{\partial^2 \pi}{\partial p \partial w_1} & \dfrac{\partial^2 \pi}{\partial p \partial w_2} \\ \vdots & & \vdots \\ & \cdots & \end{bmatrix} = \begin{bmatrix} \dfrac{\partial y(p,w)}{\partial p} & \cdots \\ \vdots & \\ & \cdots \end{bmatrix}$$

is positive semidefinite. Since the diagonal elements must be non-negative, $\dfrac{\partial y(p,w)}{\partial p} \geq 0$.

(c) The FOC say $p \dfrac{\partial f}{\partial x_1} = w_1$ and $p \dfrac{\partial f}{\partial x_2} = w_2$

Write the input demands as $x_1(p)$ add $x_2(p)$; note that these also depend on w but we are holding the input prices constant. Plug the demands back into the FOC, which yields the identities

$$p \frac{\partial f[x_1(p), x_2(p)]}{\partial x_1} \equiv w_1 \text{ and } p \frac{\partial f[x_1(p), x_2(p)]}{\partial x_2} \equiv w_2$$

or more simply

$$p f_1[x_1(p), x_2(p)] \equiv w_1$$
$$p f_2[x_1(p), x_2(p)] \equiv w_2$$

Differentiate everything with respect to p, which gives

$$f_1 + p f_{11} x_1' + p f_{12} x_2' = 0$$
$$f_2 + p f_{21} x_1' + p f_{22} x_2' = 0$$

Putting this in matrix form: $p\begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22}\end{bmatrix}\begin{bmatrix} x_1' \\ x_2'\end{bmatrix} = \begin{bmatrix} -f_1 \\ -f_2\end{bmatrix}$

Assuming the Hessian is negative definite, we have

$$\begin{bmatrix} x_1' \\ x_2'\end{bmatrix} = \frac{1}{p}\begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22}\end{bmatrix}^{-1}\begin{bmatrix} -f_1 \\ -f_2\end{bmatrix}$$

Now think about output: $y(p) = f[x_1(p), x_2(p)]$

So $y'(p) = f_1 x_1' + f_2 x_2' = -\frac{1}{p}\begin{bmatrix} f_1 & f_2\end{bmatrix}\begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22}\end{bmatrix}^{-1}\begin{bmatrix} f_1 \\ f_2\end{bmatrix}$

Because the Hessian is negative definite its inverse is also negative definite, and $h\begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22}\end{bmatrix}^{-1} h < 0$ for all $h \neq 0$.

This implies $y'(p) > 0$ so the firm produces more output when ~~output~~ output price rises.

3 (a) We need to show that $c(tw, y) = tc(w,y)$, for $t > 0$.

Let x be cost min for prices w. We want to show that x is also cost min for prices (tw). Suppose it is not. Then there is some x' such that $(tw)x' < (tw)x$. But $t > 0$ implies $wx' < wx$. This contradicts the optimality of x at the prices w. Therefore x must be optimal both for (tw) and w. So $c(tw,y) = (tw)x = tc(w,y)$.

(b) Shepherd's Lemma says that $\frac{\partial c(w,y)}{\partial w_i} = x_i(w,y)$ for $i = 1 \ldots n$ where $x_i(w,y)$ is the conditional input demand for i.

To prove this, suppose x^* is an optimal input vector when the prices are w^*.

Define $g(w) = c(w,y) - wx^* \leq 0$
where the inequality follows from the fact that x^* may not minimize cost at the prices w. This function has a maximum value of zero at $w = w^*$ because $c(w^*, y) = w^* x^*$. Therefore the FOC for a max hold at w^* $\Rightarrow$ $\frac{\partial g(w^*)}{\partial w_i} = \frac{\partial c(w^*, y)}{\partial w_i} - x_i^* = 0$ $(i = 1 \ldots n)$

Since this is true for any w^* and x^*, we have

$$\frac{\partial c(w,y)}{\partial w_i} = x_i(w,y) \text{ for all } i = 1 \ldots n.$$

(c) From Shephard's Lemma, $\frac{\partial c(w,y)}{\partial w} = \left[\frac{\partial c(w,y)}{\partial w_1} \ldots \frac{\partial c(w,y)}{\partial w_n}\right]$

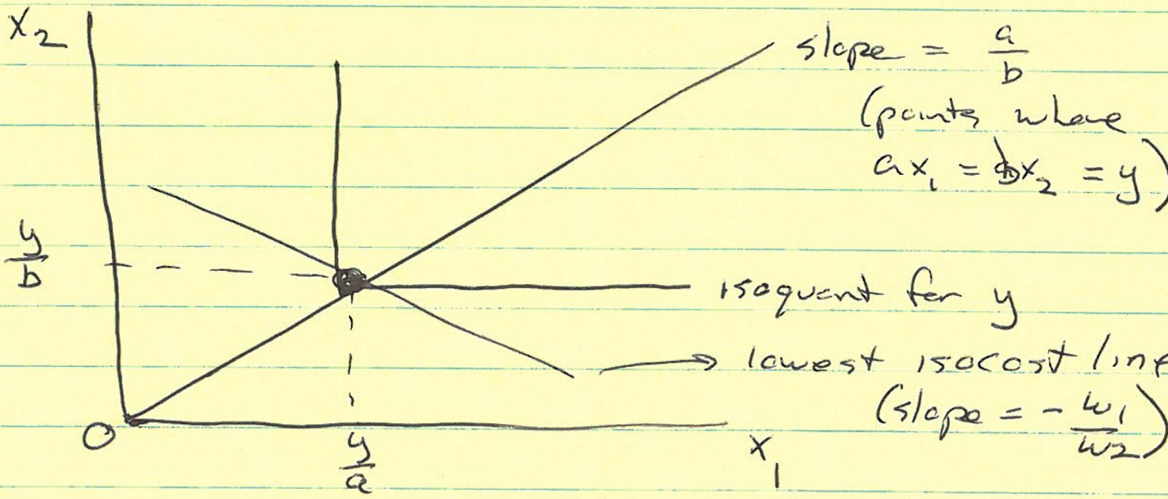
[Another way to get the result is to use Euler's Theorem and linear homogeneity]

$$= \left[x_1(w,y) \ldots x_n(w,y)\right]$$

So the scalar product of the two vectors is

$$\frac{\partial c(w,y)}{\partial w} \cdot w = x(w,y) \cdot w = \sum_{i=1}^{n} w_i x_i(w,y) = c(w,y).$$

4 (a)



The cost min solution is $x_1 = \frac{y}{a}$ and $x_2 = \frac{y}{b}$ so the cost function is $c(w,y) = w_1\left(\frac{y}{a}\right) + w_2\left(\frac{y}{b}\right)$

$$= y\left[\frac{w_1}{a} + \frac{w_2}{b}\right]$$

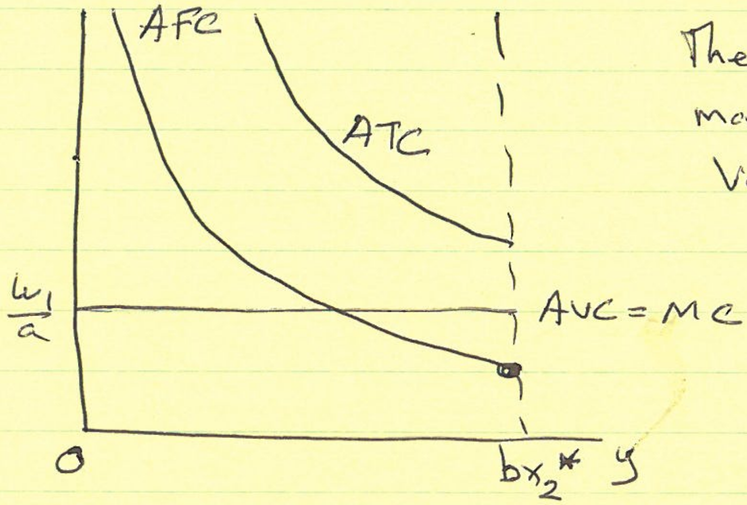
4 (b) in the short run $y = \min\{ax_1, bx_2^*\}$

For $ax_1 \le bx_2^*$ or $x_1 \le \frac{bx_2^*}{a}$ we have $y = ax_1$

For $ax_1 \ge bx_2^*$ or $x_1 \ge \frac{bx_2^*}{a}$ we have $y = bx_2^*$

So

$$c(w, y, x_2^*) = w_1\left(\frac{y}{a}\right) + w_2 x_2^* \quad \text{for } 0 \le y \le bx_2^*.$$



The firm cannot produce more than bx_2^*.

Variable cost is $\frac{w_1 y}{a}$

So $AVC = \frac{w_1}{a}$ which is a constant.

$MC = \frac{\partial c}{\partial y} = \frac{w_1}{a}$.

So AVC = MC are both horizontal.

ATC is the sum of AFC + AVC.

Since $AFC = \frac{w_2 x_2^*}{y}$ is decreasing

and AVC is constant, ATC must be decreasing.

(c) From part (a), $LAC = \frac{w_1}{a} + \frac{w_2}{b}$ = constant (horizontal).

Also from part (a), $x_2^* = \frac{y^*}{b}$

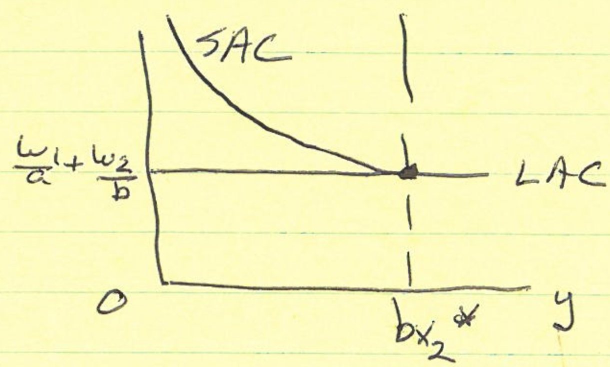
From part (b), $SAC = \frac{w_1}{a} + \frac{w_2 x_2^*}{y}$

This is > LAC for $y < y^*$

and = LAC at $y = y^* = bx_2^*$

which is the largest possible output given x_2^*.

So SAC is falling and meets LAC at $y = bx_2^*$.



5 (a). FOC $\Rightarrow$ $w_1 = \lambda f_1 = \lambda a(x_1x_2)^{a-1}x_2$
$$w_2 = \lambda f_2 = \lambda a(x_1x_2)^{a-1}x_1$$

Divide one equation by the other to get $\frac{w_1}{w_2} = \frac{x_2}{x_1}$ or $\frac{x_1}{x_2} = \left(\frac{w_1}{w_2}\right)^{-1}$.

We have $\sigma = -\frac{\partial\left(\frac{x_1}{x_2}\right)}{\partial\left(\frac{w_1}{w_2}\right)} \cdot \frac{\left(\frac{w_1}{w_2}\right)}{\left(\frac{x_1}{x_2}\right)} = \frac{\left(\frac{w_1}{w_2}\right)^{-2} \cdot \left(\frac{w_1}{w_2}\right)}{\left(\frac{w_1}{w_2}\right)^{-1}} = +1$

This implies that the fraction of total cost spent on each input remains constant no matter what happens to $\frac{w_1}{w_2}$ when y is held fixed. Consider the ratio of expenditures $\frac{w_1x_1}{w_2x_2} = \left(\frac{w_1}{w_2}\right)\left(\frac{x_1}{x_2}\right)$. With an elasticity $= 1$, a 1% increase in $\left(\frac{w_1}{w_2}\right)$ yields a 1% decrease in $\left(\frac{x_1}{x_2}\right)$ so the expenditure ratio won't change.

(Note: This is a Cobb-Douglas production function and the symmetry of the function implies that each input is paid ½ of total cost)

(b) $e(x) = \frac{df(tx)}{dt} \cdot \frac{t}{f(x)}\Big|_{t=1} = \frac{d}{dt}\left[t^k f(x)\right] \cdot \frac{t}{f(x)}\Big|_{t=1}$

$$= kt^{k-1}f(x)\frac{t}{f(x)}\Big|_{t=1} = k.$$

This says that the elasticity with respect to scale is equal to the constant k (globally). We know that $e(x^*) < 1 \Rightarrow$ LAC rising at y^* etc. So if $k < 1$ (DRS) LAC is always rising; if $k = 1$ (CRS) then LAC is horizontal; and if $k > 1$ (IRS) then LAC is always falling.

5 (c). (i) If $f(x)$ is concave then any x^* satisfying the FOC actually does solve the problem although the solution may not be unique.

(ii) If $f(x)$ is strictly concave then an x^* that satisfies the FOC must be the <u>unique</u> solution.

(iii) if the Hessian is negative definite everywhere then an x^* that satisfies the FOC must be the unique solution <u>and</u> we can use the implicit function theorem to treat x^* as a differentiable function of the prices.

<u>Note:</u> This implies that you do not need to check SOC (necessary or sufficient).